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THE NOISE REJECTION, TRACKING LAGS, AND TRANSIENT DECAY FOR EXP--ETC(U)
SEP 77 J HAYWARD

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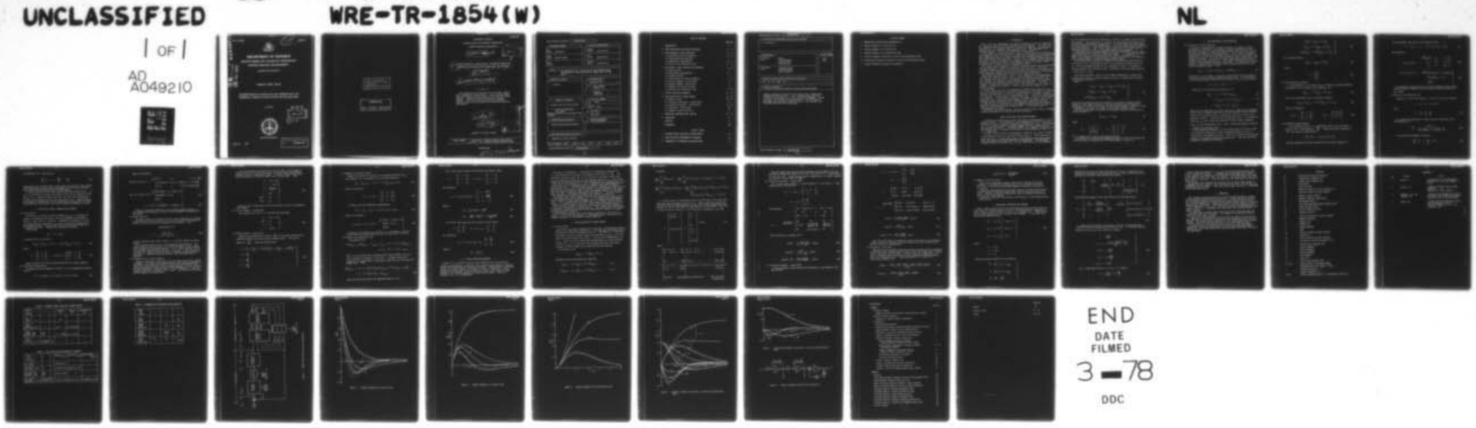

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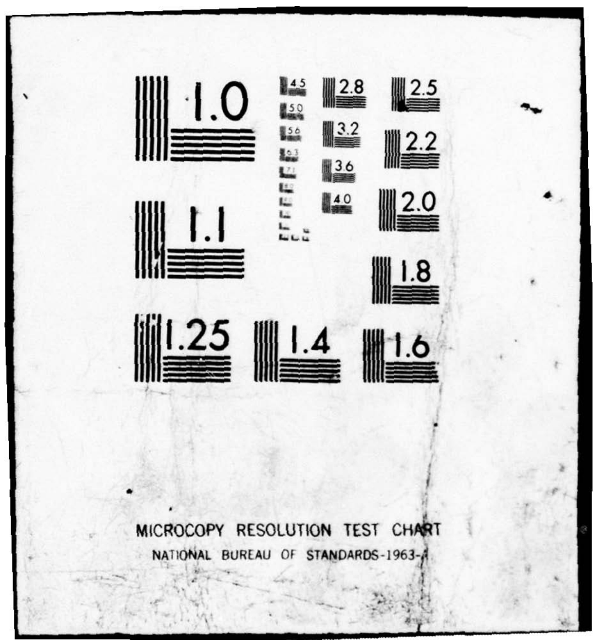
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THE NOISE REJECTION, TRACKING LAGS, AND TRANSIENT DECAY FOR EXPONENTIAL TRACKING FILTERS OF ORDER, ONE, TWO AND THREE

J. HAYWARD

DEFENCE SCIENCE AND TECHNOLOGY ORGANISATION

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SUMMARY

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Basic equations are derived for the third order, sampled data, exponential tracking filter. Lags, transient decay and noise rejection are calculated for critical and optimal damping. Graphs and tables are provided for design purposes. The third order continuous data (analogue) tracking filter is also analysed, and equivalence relations are derived.

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LIST OF FIGURES

1. INTRODUCTION

Most, if not all, measurements of a physical quantity, involve the comparison of a reference quantity, of known magnitude, with the unknown. As a result of the comparison, the reference quantity is adjusted in magnitude, and the process repeated until the comparison shows negligible error. The magnitude of the reference quantity is then the estimated value of the unknown.

If the unknown is varying with time, it may be that the variation within one intercomparison period is sufficient to prevent the achievement of a satisfactorily small comparison error with simple adjustment of the reference quantity; that is the system suffers from a lag error. In this case, an improved estimate may be obtained if the rate of change of the unknown is also estimated, and an additional adjustment is made to the value of the reference, equal to the product of the estimated rate of change, and the intercomparison period.

This process can be extended to include an estimate of higher order derivatives, each one being used to adjust the estimate of the lower ones. Apart from increased complexity, penalties include the greater sensitivity to random errors occurring in the comparison process, and the need for a succession of comparisons before reasonable estimates are obtained of the higher derivatives.

The process described above is often called a sampled data tracking system. It breaks down logically into two parts: the discriminator, which compares the last estimate with the variable, and estimates the correction required; and the tracking filter-predictor, which remembers the last value of the estimates, and calculates the improved estimates for the next comparison. The whole comprises a sampled data feedback controller, which exhibits the usual stability problems, and transient responses to initial conditions and changes in the primary variable.

In this report, the third order exponential tracking filter-predictor, and the system incorporating it, are defined and analysed in some detail, to determine the basic characteristics of stability, transfer function, transient response, tracking lags, and sensitivity to discriminator noise. Lower order systems are also examined. A brief analysis along the same lines, is made of a third order, continuous data, tracking filter-predictor, an example of which is the electromechanical range or bearing tracker, commonly used in older radar systems. It is shown that the characteristics of the sampled and continuous data systems are almost identical, providing the inter-sampling period is much less than the effective time constants of the tracking filter. An equivalence table of system parameters is given.

Familiarity with matrix notation, and with classical control theory is assumed. Statistical parameters are derived from first principles.

2. MODEL DEFINITIONS AND PREDICTION MATRIX

A sampled data tracking loop, using a third order filter-predictor is illustrated in figure 1. The variable, x, is the unknown, and $\hat{x}$, $\hat{\dot{x}}$ and $\hat{\ddot{x}}$ are the estimates of x and its first two derivaties with respect to time. The suffixes, t-1, t, t+1, denote successive sampling (comparison) times, separated by the inter-comparison period, T. The double suffix, t/t-1, denotes an estimate for the value at time, t, calculated on the basis of data available at time, t-1, i.e. a prediction.

At time, t, there exists an (unobservable) correction, x_{ct}, which, if added to the predicted value, $\hat{x}_t$, will give the true value, x_t. The correction is mixed with noise, or other random errors in the Physical System under examination, which may also add non-linearities and dispersions. The Correction Estimator produces an estimate, $\hat{x}_{ct}$, of x_{ct}, which contains a noise component,

δ_t, and a multiplicative error coefficient, D, which is a function of the correction required. This coefficient, D, accounts for non-linearities and gain errors in the overall Discriminator characteristic. It ideally takes the value, unity, but always exhibits an S-type characteristic, due to the finite range of sensitivity of the Discriminator. (This finite range is necessary if two or more physical quantities must be resolved, e.g. two radar echoes, closely spaced.)

A Sampling Device is located somewhere within the Discriminator section, and is shown ahead of the Correction Estimator in figure 1. In the case of a pulsed radar, this function is provided by the pulse itself. Its purpose is to staticise the operation of the Estimator, at values appropriate to the sampling time, ignoring changes in variables between sampling times.

The Tracking Filter combines the estimates of the physical quantity and its derivatives, $\hat{x}$, $\hat{\dot{x}}$, $\hat{\ddot{x}}$, applicable at time, t, but predicted from data from time, t-1, with the estimated correction, $\hat{x}_{ct}$, applicable at time, t, to produce a revised estimate of $\hat{x}$, $\hat{\dot{x}}$ and $\hat{\ddot{x}}$ at time, t, and based on the extra data available at that time (i.e. $\hat{x}_{ct}$).

The Tracking Predictor carries out the simple extrapolation to predict the values of $\hat{x}$, $\hat{\dot{x}}$, and $\hat{\ddot{x}}$, at the next sampling instant, T seconds away, and based on information at time, t.

Thus:

$$\left.\begin{aligned}
\hat{x}_{t+1/t} &= \hat{x}_{t/t} + T.\,\hat{\dot{x}}_{t/t} + 0.5\,T^2.\,\hat{\ddot{x}}_{t/t} \\
T.\hat{\dot{x}}_{t+1/t} &= T.\,\hat{\dot{x}}_{t/t} + T^2.\,\hat{\ddot{x}}_{t/t} \\
0.5T^2.\hat{\ddot{x}}_{t+1/t} &= 0.5\,T^2\,\hat{\ddot{x}}_{t/t}
\end{aligned}\right\} \qquad (1)$$

Note that the derivaties have been expressed as equivalent distances covered during the inter-sampling period, T, and that the predicted acceleration is unchanged over T, since a third order system has been assumed. Equation (1) can be extended or contracted for higher or lower order tracking filters.

Matrix notation proves very useful for subsequent analysis, and equation (1) can be expressed as:

$$\hat{X}_{t+1/t} = P.\,\hat{X}_{t/t} \qquad (2)$$

where:

$$\hat{X} = \begin{bmatrix} \hat{x} \\ T.\hat{\dot{x}} \\ 0.5T^2\hat{\ddot{x}} \end{bmatrix} \;; \qquad P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3)$$

$\hat{X}$ is a column vector, and P is the prediction matrix.

The Register is clocked between sampling instants, to hold the prediction for the next correction estimation and filtering procedure, at time, t+1.

3. THE EXPONENTIAL FILTER-PREDICTOR

3.1 Derivation of filter equations

Consider the measurement of a physical quantity, x, subject to noise. The estimate is improved if the average of several measurements is taken, providing the noise is uncorrelated between samples, and the value of x does not change significantly, during the sequence. If the value is changing rather slowly, the earlier measurements are of less use than later ones, for estimating the current value, and should therefore receive less weight in the average calculation. One possible estimator gives equal weight to the last N samples, and zero to earlier ones. This requires storage of the last N values, so that subsequent estimates can be made. The exponential estimator weights all samples by a factor which reduces exponentially with the time since the sample was taken. Thus:

$$\hat{x}_{t/t} = a.\{x_t + (1-a).x_{t-1} + (1-a)^2.x_{t-2} + \dots\dots\} \tag{4}$$

The factor, a, lies between one and zero, and determines how much weight is given to the earlier samples, compared with later ones. The coefficient, a, outside the brackets, normalises the estimate, as can be seen by putting

$$x_t = x_{t-1} = x_{t-2} = \dots\dots$$

Equation (4) can be expressed recursively, as :

$$\hat{x}_{t/t} = a.\,x_t + (1-a).\,\hat{x}_{t-1/t-1} \tag{5}$$

$$= \hat{x}_{t-1/t-1} + a.(x_t - \hat{x}_{t-1/t-1}) \tag{6}$$

$$= \hat{x}_{t/t-1} + a.\,(x_t - \hat{x}_{t/t-1}) \tag{7}$$

Equation (7) follows from (6) if the value of x is assumed to be unchanging. If the value is changing, equation (7) intuitively gives a better estimate than (6) (or the same estimate, if derivatives of $\hat{x}$ are not used in the prediction process). Note that the filtered value, $\hat{x}_{t/t}$, is the predicted value, $\hat{x}_{t/t-1}$, plus a times the difference between the actual measured value, x_t, and the predicted value. Thus, a times the measured correction is added to the predicted value.

By an extension of equation (7), the filtered values of $T.\hat{\dot{x}}$ and $0.5T^2.\hat{\ddot{x}}$ are obtained by adding β and γ times the measured correction, $\hat{x}_{ct}$, as corrections to the predicted values. (This is the reason for representing, $\hat{\dot{x}}$ and $\hat{\ddot{x}}$, as equivalent distances in equation (1).) Thus the third order exponential tracking filter becomes:

$$\left.\begin{aligned} \hat{x}_{t/t} &= \hat{x}_{t/t-1} + \alpha.\ \hat{x}_{ct} \\ T.\ \hat{\dot{x}}_{t/t} &= T.\ \hat{\dot{x}}_{t/t-1} + \beta.\ \hat{x}_{ct} \\ 0.5T^2.\ \hat{\ddot{x}}_{t/t} &= 0.5T^2.\ \hat{\ddot{x}}_{t/t-1} + \gamma.\ \hat{x}_{ct} \end{aligned}\right\} \tag{8}$$

Or in matrix notation:

$$\hat{X}_{t/t} = X_{t/t-1} + F.\ \hat{x}_{ct} \tag{9}$$

where:

$$F = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \tag{10}$$

3.2 Filter-predictor equations

By using equation (2) to eliminate $\hat{X}_{t/t}$, we obtain the equation defining the operation of the combined filter-predictor, via:

$$\hat{X}_{t+1/t} = P.\ \hat{X}_{t/t-1} + P.F.\ \hat{x}_{ct} \tag{11}$$

This can be rearranged to give a difference equation:

$$\hat{X}_{t+1/t} - \hat{X}_{t/t-1} = (P-I).\ \hat{X}_{t/t-1} + P.F.\ \hat{x}_{ct} \tag{12}$$

where, by evaluation:

$$(P-I) = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}; \qquad P.F. = \begin{bmatrix} \alpha + \beta + \gamma \\ \beta + 2\gamma \\ \gamma \end{bmatrix} \tag{13}$$

3.3 Differential approximation

If the sampling interval, T, is sufficiently small, the left side of equation (12) is approximately, $T.\ \frac{d\hat{X}}{dt}$. Taking the Laplace transform of (12), and using this approximation:

$$T.\ (s.\bar{\hat{X}} - \hat{X}_0) \approx (P-I).\ \bar{\hat{X}} + P.F.\ \bar{\hat{x}}_c$$

where $\hat{X}_0$ represents the initial conditions of the vector variable, $\hat{X}$.

By rearrangement, and using the left inverse notion:

$$\hat{X} \approx \left\{ (T.s + 1).\ I - P \right\}^{-1} . \left\{ P.F.\ \bar{\hat{x}}_c + T.\ \hat{X}_0 \right\} \tag{14}$$

By evaluation:

$$\left\{ (T.s+1).I-P \right\}^{-1} = \frac{1}{T^3.s^3} \begin{bmatrix} T^2.s^2 & T.s & 2 + T.s \\ 0 & T^2.s^2 & 2\ T.s \\ 0 & 0 & T^2.s^2 \end{bmatrix} \tag{15}$$

$$\left\{ (T.s+1).I-P \right\}^{-1}.\ P.F = \frac{1}{T^3.s^3} \begin{bmatrix} T^2.s^2.(\alpha+\beta+\gamma) + T.s.(\beta+3\gamma)+2\gamma \\ T^2.s^2.(\beta+2\gamma) + 2\ T.s.\gamma \\ T^2.s^2.\gamma \end{bmatrix} \tag{16}$$

An approximate closed relation for $\hat{X}$ can be obtained from the inverse Laplace transform of equation (14), providing the Laplace transform of $\hat{x}_c$ is known.

3.4 Velocity and acceleration lags

Suppose $\hat{x}_c$ is driven (by feedback) to cause $\hat{x}$ to follow the response:

$$\hat{x} = x_0 + v_0.t + 0.5\cdot a_0.t^2$$

Then:

$$\bar{\hat{x}} = \frac{x_0}{s} + \frac{v_0}{s^2} + \frac{a_0}{s^3} \tag{17}$$

By inverting the approximate transfer function relating $\hat{x}_c$ and $\hat{x}$, *from equation* (14):

$$\bar{\hat{x}}_c \approx \frac{T^3.s^3.\ \bar{\hat{x}}}{T^2.s^2.(\alpha+\beta+\gamma) + T.s.(\beta+3\gamma) + 2\gamma} \tag{18}$$

The steady state response is given by:

$$\lim_{t\to\infty} \hat{x}_c = \lim_{s\to 0} s.\ \bar{\hat{x}}_c \tag{19}$$

From equations (17), (18) and (19):

$$\lim_{t\to\infty} \hat{x}_c = 0 \;;\; \frac{T^2.a_0}{\beta} \;;\; \frac{T.v_0}{\alpha} \qquad (20)$$

respectively, for the third order, second order (γ=o) and first order (γ=β=o) filter-predictors, showing a finite acceleration lag for the second order, and a finite velocity lag for the first order case.

A similar conclusion can be drawn from the difference equation (12). If $\hat{x}_{ct}$ is to reach the zero value, the change in $\hat{X}$ can only be supplied via the matrix (P-I). Where this matrix has an all zero row, the corresponding element of the difference vector, $\Delta\,\hat{x}$, must also be zero. Hence we require zero changes in acceleration, velocity and position, respectively for third, second and first order filters, if no lag in position tracking is to occur.

4. NOISE FREE, LINEAR, TRACKING LOOP RESPONSE

4.1 Exact analysis

From figure 1, we want to find the response, $\hat{X}_{t/t-1}$, to a sequence of inputs, x_t, assuming no noise and a linear discriminator characteristic. Thus δ_t is zero, and D is a constant. Examination of equations (11) and (13) shows that the coefficients, α, β and γ are directly multiplied by D, and therefore there is no loss of generality if the non-unity value of D is absorbed into them. Therefore D is assumed to be unity.

Substituting:

$$\hat{x}_{ct} = x_t - \hat{x}_{t/t-1}$$

in equation (12), we find that:

$$\hat{X}_{t+1/t} - \hat{X}_{t/t-1} = (P - I - P.F_1).\,\hat{X}_{t/t-1} + P.F.\,x_t \qquad (21)$$

where:

$$F_1 = \begin{bmatrix} \alpha & 0 & 0 \\ \beta & 0 & 0 \\ \gamma & 0 & 0 \end{bmatrix} \;;\quad (P-I-P.F_1) = \begin{bmatrix} -\alpha-\beta-\gamma & 1 & 1 \\ -\beta-2\gamma & 0 & 2 \\ -\gamma & 0 & 0 \end{bmatrix} \qquad (22)$$

Again, equation (21) can be evaluated directly by a digital computer.

4.2 Differential approximation

By analogy with the procedure of section 3.3, we can approximate equation (21) by:

$$\bar{\hat{X}} \approx \{(T.s+1).I - P + P.F_1\}^{-1}.\{P.F.\,\bar{x} + T.\hat{X}_0\} \qquad (23)$$

where, by evaluation:

$$\{(T.s+1).I-P+P.F_1\}^{-1} = \frac{1}{\Delta}\begin{bmatrix} T^2.s^2 & T.s & 2+T.s \\ -T.s.(\beta+2\gamma)-2\gamma & T^2.s^2+T.s.(\alpha+\beta+\gamma)+\gamma & 2\,T.s+2\alpha+\beta \\ -T.s.\gamma & -\gamma & T^2s^2+T.s(\alpha+\beta+\gamma)+\beta+2\gamma \end{bmatrix} \tag{24}$$

$$\{(T.s+1).I-P+P.F_1\}^{-1}P.F = \frac{1}{\Delta}\begin{bmatrix} T^2.s^2.(\alpha+\beta+\gamma) + T.s.(\beta+3\gamma) + 2\gamma \\ T^2.s^2.(\beta+2\gamma) + T.s.2\gamma \\ T^2.s^2.\gamma \end{bmatrix} \tag{25}$$

$$\Delta = T^3.s^3 + T^2.s^2.(\alpha+\beta+\gamma) + T.s.(\beta+3\gamma) + 2\gamma \tag{26}$$

Clearly, Δ, being the denominator of the system transfer function, is the characteristic function, the roots of which determine the stability of the system.

4.3 System stability

The Routh stability criterion(ref.1) for a third order function, reduces to the condition that the product of the two inner coefficients exceeds the product of the two outer ones (assuming all are positive). Thus:

$$(\alpha+\beta+\gamma)(\beta+3\gamma) > 2\gamma$$

$$\therefore\ \gamma < \frac{\beta\ (\alpha+\beta)}{2-3\alpha-4\beta-3\gamma} \tag{27}$$

For the second and first order systems, $\gamma = 0$, and the systems are always stable.

This criterion applies only to the differential approximation. The actual system will be less stable, the longer the period, T, due to phase shifts produced by the sampling process. If T is small compared with the *time constants* associated with the roots of (26) (i.e. $\alpha \ll 1$, see section 5), and if the stability margin from (27) is adequate, further analysis is unwarranted. Otherwise an analysis based on the z-transform(ref.2) is required.

4.4 Coefficients for critical damping

Being of third order, Δ has three roots, one being real and two, complex conjugates. Classical control theory suggests that optimum performance (in the sense of best compromise between noise rejection and transient decay time) occurs when the complex roots are damped somewhat less than critically, but that performance is not much worse at critical damping (i.e. when the imaginary part of the conjugate roots is zero). For simplicity of analysis we will examine the critical case.

If the time constant associated with the third root, is much larger or smaller than that of the first two roots, the system tends to approximate towarded a first or second order system. To obtain a typical third order system, therefore, we choose all three roots, real and equal. Thus:

$$\left.\begin{aligned} \Delta &= (T.s + \rho)^3 \\ \alpha &= 1 - (1-\rho)^3 \\ \beta &= \frac{3\rho^2(2-\rho)}{2} \\ \gamma &= \rho^3/2 \\ \alpha+\beta+\gamma &= 3\rho \\ \beta+3\gamma &= 3\rho^2 \end{aligned}\right\} \qquad (28)$$

Naturally (28) satisfies the stability criterion (27) with a considerable safety margin.

4.5 Critical damping - second order

From equation (26), with $\gamma = 0$, and roots real and equal

$$\left.\begin{aligned} \Delta &= (T.s + \rho)^2 \\ \alpha+\beta &= 2\rho \\ \beta &= \rho^2 \\ \alpha &= \rho(2-\rho) \end{aligned}\right\} \qquad (29)$$

4.6 Optimum damping - second order

Benedict and Bordner(ref.3) *choose* $\beta = \frac{\alpha^2}{2-\alpha}$ as the value giving the best compromise between noise rejection and transient decay. This gives a damping of $\frac{1}{\sqrt{2-\alpha}}$ times the critical value.

$$\left.\begin{aligned} \Delta &= \left\{T.s + \frac{\alpha}{2-\alpha}\left(1 + j\sqrt{1-\alpha}\right)\right\}\left\{T.s + \frac{\alpha}{2-\alpha}\left(1 - j\sqrt{1-\alpha}\right)\right\} \\ \rho &= \frac{\alpha}{2-\alpha} \\ \alpha &= \frac{2\rho}{1+\rho} \\ \beta &= \frac{2\rho^2}{1+\rho} \end{aligned}\right\} \qquad (30)$$

4.7 Response of filtered variable

Equation (21) gives the response of the predicted variable, $\hat{X}_{t/t-1}$. Using equation (2), the response of the filtered variable is:

$$\hat{X}_{t/t} - \hat{X}_{t-1/t-1} = (P - I - F_1 .P).\hat{X}_{t-1/t-1} + F.x_t \tag{31}$$

where, by evaluation:

$$(P - I - F_1 .P) = \begin{bmatrix} -\alpha & 1-\alpha & 1-\alpha \\ -\beta & -\beta & 2-\beta \\ -\gamma & -\gamma & -\gamma \end{bmatrix} \tag{32}$$

As before, with the differential approximation:

$$\bar{\hat{X}} \approx \{(T.s + 1).I - P + F_1 .P\}^{-1}. \{F.\bar{x} + T.\hat{x}_0\} \tag{33}$$

where, by evaluation:

$$\{(T.s+1).I - P + F_1 .P\}^{-1}.F = \frac{1}{\Delta}\begin{bmatrix} T^2.s^2.\alpha + T.s(\beta+\gamma) + 2\gamma \\ T^2.s^2.\beta + 2T.s.\gamma \\ T^2.s^2.\gamma \end{bmatrix} \tag{34}$$

Naturally, the characteristic function, Δ, is unchanged. Note the trivial difference between equations (25) and (34), with $\gamma \ll \beta \ll \alpha$.

4.8 Velocity and acceleration lags

From equation (21):

$$(X-\hat{X})_{t+1/t} - (X-\hat{X})_{t/t-1} = X_{t+1/t} - X_{t/t-1} + (P - I - P.F_1).(X-\hat{X})_{t/t-1}$$
$$- P.F.\ x_t - (P - I - P.F_1).\ X_{t/t-1}$$

where X is the column vector representing the input variable, and $X_{t+1} = P_x.X_t$ (analogous to equations (3) and (2)). Assuming steady state conditions are reached after a sufficiently large time, the left hand side becomes zero. Thus:

$$(X-\hat{X})_{t/t-1} = (I - P + P.F_1)^{-1}. \left\{(P_x-P).X_{t/t-1} + P.(F_1 .X_{t/t-1} - F.x_t)\right\}$$
$$= (I - P + P.F_1)^{-1}. (P_x - P).X_{t/t-1} \tag{35}$$

since the second term within the compound brackets is zero.

For a second order processor with constant acceleration input:

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} ; \quad \gamma = 0; \; P_x = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

By evaluation:

$$(I - P + P.F_1)^{-1}.(P_x - P) = \begin{bmatrix} 0 & 0 & 2/\beta \\ 0 & 0 & \frac{2\alpha+\beta}{\beta} \\ 0 & 0 & 1 \end{bmatrix} \tag{36}$$

Whence:

$$x - \hat{x} = \frac{2}{\beta}. \, 0.5 a_0 T^2 = \frac{a_0 . T^2}{\beta} \tag{37}$$

$$\dot{x} - \hat{\dot{x}} = \frac{2\alpha+\beta}{\beta}. \, \frac{0.5.a_0.T^2}{T} = \frac{a_0.T.(2\alpha+\beta)}{2\beta} \tag{38}$$

For a first order processor with constant velocity input:

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} ; \quad \beta = \gamma = 0; \; P_x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

By evaluation:

$$(I - P + P.F_1)^{-1}.(P_x - P) = \begin{bmatrix} 0 & \frac{1}{\alpha} \\ 0 & 1 \end{bmatrix} \tag{39}$$

Whence:

$$x - \hat{x} = \frac{1}{\alpha}.T.v_0 \tag{40}$$

5. TYPICAL TRANSIENT RESPONSES

Figures 2 to 6 inclusive illustrate the transient decay of the position, velocity and acceleration corrections $(X-\hat{X})$, following step changes in position, velocity and acceleration of the measured variable. Equation (21) is evaluated directly by digital computer, for values of α, β and γ, from Table 1, being appropriate to equal root critical damping for first, second and third order trackers, and to optimum damping for a second order tracker. These are labelled, 1, 2C, 3, 2_0 respectively.

The curves are normalised. The abscissa is calibrated in terms of $^{t}/\tau$, i.e. time normalised to the basic time constant of the filter ($\tau = \frac{T}{\rho}$). The normalised sampling interval ($\frac{T}{\tau} = \rho$) is 0.02 for the continuous curves, which are indistinguishable from curves for the limiting case of $\rho = 0$. The crosses (shown only for the first and third order trackers) mark the responses for $\rho = 0.2$.

The locus of the response for $\rho = 0.2$ is reasonably well approximated by the limiting curve ($\rho = 0$), differing mainly in greater overshoots and slightly shorter decay times. The first order tracker has a good recovery from a position step (figure 2); tolerates a velocity step with a lag, providing the lag error does not exceed the drop out range of the discriminator (figure 3); and rapidly loses lock under a prolonged acceleration step (figure 4). It gives the best noise rejection on the position estimate (see Section 6, and Table 2).

The second order tracker handles a velocity step without lag, (figure 3) and an acceleration step with lag (figure 4). In all aspects the optimum damping case is superior to the critically damped case (i.e. reduced lag and recovery time). It is comparable with the first order system, under a step in position. It gives a velocity estimate, but with a noisier position estimate (Table 2).

The third order system is inferior in all respects, except for providing an acceleration estimate and no acceleration lag. The recovery time could perhaps be improved by separating the roots of the characteristic equation, to reduce the long overshoot (figure 2).

The lags check with the results calculated in Section 4.6, and listed in Table 1. For a given time constant, τ, and damping factor, the lags are only slightly dependent on the normalised sampling time (ρ).

6. NOISE REJECTION BY LINEAR TRACKER

6.1 General theory

We consider the linear exponential tracking loop, illustrated in figure 1, with D = 1, as discussed in Section 4.1. In this case, the external signal, is arbitrary, and will be assumed to be zero. The noise, δ_t, is non-zero. (Note that it does not include output noise fed back via $\hat{x}_{t/t-1}$, since this latter term appears explicitly at $\hat{x}_{ct}$). Thus δ_t can be considered to act at x_t. Hence, we wish to obtain the standard deviation of the predicted variable, $\hat{X}_{t+1/t}$, as a function of the standard deviation of the input variable, x_t, after steady state conditions have been achieved.

From equation (21):

$$\hat{X}_{t+1/t} = Q.\ \hat{X}_{t/t-1} + R.\ x_t$$

Writing the first and second rows, explicity:

$$\hat{x}_{t+1/t} = q_{11}.\ \hat{x}_{t/t-1} + q_{12}.T.\hat{\dot{x}}_{t/t-1} + \ldots \qquad + r_1.x_t$$

$$T.\hat{\dot{x}}_{t+1/t} = q_{21}.\ \hat{x}_{t/t-1} + q_{22}.T.\hat{\dot{x}}_{t/t-1} + \ldots \qquad + r_2.x_t$$

Therefore:

$$\lim_{N\to\infty} \sum_{t=0}^{N} \frac{1}{N} (\hat{x}_{t+1/t})^2 = \lim_{N\to\infty} \sum_{t=0}^{N} \frac{1}{N} \{ q_{11}.\hat{x}_{t/t-1} + q_{12}.T.\hat{x}_{t/t-1} + \ldots + r_1.x_t \}^2$$

$$\lim_{N\to\infty} \sum_{t=0}^{N} \frac{1}{N} (\hat{x}_{t+1/t}.T.\hat{x}_{t+1/t}) = \lim_{N\to\infty} \sum_{t=0}^{N} \frac{1}{N} \{ q_{11}.\hat{x}_{t/t-1} + q_{12}.T.\hat{x}_{t/t-1} + \ldots + r_1.x_t \}. \{ q_{21}.\hat{x}_{t/t-1} + q_{22}.T.\hat{x}_{t/t-1} + \ldots + r_2.x_t \}$$

It is clear that x_t is uncorrelated with all other terms on the right hand side, since they are based on data at (t-1). Under steady state conditions, the limit on the left hand side has the same value as the corresponding limit product term on the right hand side (ignoring the q coefficients). Thus, for a system of order M, we have $(M + C_2^M)$ equations, expressable in the form:

$$V. \begin{bmatrix} \mathrm{Var}(\hat{x}) \\ \mathrm{Var}(T.\hat{x}) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \mathrm{Covar}(\hat{x},T.\hat{x}) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} = \begin{bmatrix} r_1^2 \\ r_2^2 \\ \cdot \\ \cdot \\ r_M^2 \\ r_1.r_2 \\ r_1.r_3 \\ \cdot \\ \cdot \\ \cdot \\ r_{M-1}.r_M \end{bmatrix} .\mathrm{Var}(x) \qquad (41)$$

where:

$$V = \left[\begin{array}{cccc|cccc} (1-q_{11}^2) & -q_{12}^2 & -q_{13}^2 \ldots & -q_{1M}^2 & -2q_{11}.q_{12} & -2q_{11}.q_{13} & \ldots & -2q_{1(M-1)}.q_{1M} \\ -q_{21}^2 & (1-q_{22}^2) & -q_{23}^2 \ldots & -q_{2M}^2 & -2q_{21}.q_{22} & -2q_{21}.q_{23} & \ldots & -2q_{2(M-1)}.q_{2M} \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ -q_{M1}^2 & \ldots & \ldots & (1-q_{MM}^2) & -2q_{M1}.q_{M2} & \ldots & \ldots & -2q_{M(M-1)}.q_{MM} \\ \hline -q_{11}.q_{21} & -q_{12}.q_{22} & \ldots & -q_{1M}.q_{2M} & (1-q_{11}.q_{22}-q_{12}.q_{21}) & -(q_{11}.q_{23}+q_{13}.q_{21}) & \ldots & -(q_{1(M-1)}q_{2M}+ q_{1M}.q_{2(M-1)}) \\ -q_{11}.q_{31} & \ldots & \ldots & -q_{1M}.q_{3M} & & & & \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ -q_{(M-1)}.q_{M1} & \ldots & \ldots & -q_{(M-1)M}.q_{MM} & -(q_{(M-1)1}.q_{M2}+q_{(M-1)2}.q_{M1} & \ldots & \ldots & (1-q_{(M-1)(M-1)}.q_{MM}- q_{(M-1)M}.q_{M(M-1)}) \end{array}\right] \qquad (42)$$

Thus the steady state variances and covariances of the elements of $\hat{X}$, are obtained by multiplying equation (33) by the left inverse of V, a square matrix of order $(M + C_2^{\ M})$. Evaluating the inverse, algebraically, is hardly feasible for a system of third order, or higher.

6.2 Prediction covariance - second order

For a second order system, M = 2, and the matrix, V, is of order 3. From equations (21), (22) and (13):

$$Q = P - P.F_1 = \begin{bmatrix} 1 - \alpha - \beta & 1 \\ -\beta & 1 \end{bmatrix}$$

$$R = P.F = \begin{bmatrix} \alpha + \beta \\ \beta \end{bmatrix}$$

By evaluation:

$$V = \begin{bmatrix} (2 - \alpha - \beta)(\alpha + \beta) & -1 & -2(1 - \alpha - \beta) \\ -\beta^2 & 0 & 2\beta \\ \beta(1 - \alpha - \beta) & -1 & \alpha + 2\beta \end{bmatrix}$$

$$V^{-1} = \frac{1}{\alpha\beta(4-2\alpha-\beta)} \begin{bmatrix} 2\beta & 2-\alpha & -2\beta \\ \beta^2(2-\alpha) & \begin{matrix} -\alpha^3-2\alpha^2\beta-\alpha\beta^2 \\ +2\alpha\beta+2\alpha^2+2\beta \end{matrix} & \begin{matrix} -2\beta^2(1-\alpha) \\ -2\alpha\beta(2-\alpha) \end{matrix} \\ \beta^2 & \alpha(2-\alpha)+\beta(1-\alpha) & -\beta^2 \end{bmatrix}$$

And by evaluation of equation (33):

$$\mathrm{Var}(\hat{x}) = \frac{\alpha + \beta/\alpha + \beta/2}{2 - \alpha - \beta/2} \cdot \mathrm{Var}(x) \qquad (43)$$

$$\mathrm{Var}(T.\hat{x}) = \frac{\beta^2/\alpha}{2 - \alpha - \beta/2} \cdot \mathrm{Var}(x) \qquad (44)$$

$$\mathrm{Covar}(\hat{x}, T.\hat{x}) = \frac{\beta(1 + \beta/2\alpha)}{2 - \alpha - \beta/2} \cdot \mathrm{Var}(x) \qquad (45)$$

6.3 Filtered covariance - second order

For the filtered, rather than predicted variable, we use equations (31), (32) and (10):

$$Q + (P - F_1 . P) = \begin{bmatrix} 1-\alpha & 1-\alpha \\ -\beta & 1-\beta \end{bmatrix}$$

$$R = F = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$V = \begin{bmatrix} \alpha(2-\alpha) & -(1-\alpha)^2 & -2(1-\alpha)^2 \\ -\beta^2 & \beta(2-\beta) & 2\beta(1-\beta) \\ \beta(1-\alpha) & -(1-\alpha)(1-\beta) & \alpha+2\beta-2\alpha\beta \end{bmatrix}$$

$$V^{-1} = \frac{1}{\alpha\beta(4-2\alpha-\beta)} \begin{bmatrix} \beta(2-\alpha\beta) & 2-5\alpha+4\alpha^2-\alpha^3 & 2\beta(1-\alpha)^2 \\ \beta^2(2-\alpha) & \alpha^2(2-\alpha)+2\beta(1-\alpha) & 2\beta(\beta-2\alpha+\alpha^2) \\ -\beta^2(1-\alpha) & 2\alpha-3\alpha^2+\alpha^3+\alpha\beta-\beta & \beta(4\alpha-2\alpha^2-\beta) \end{bmatrix}$$

$$\mathrm{Var}(\hat{x}) = \frac{\alpha + \beta/\alpha - 3\beta/2}{2 - \alpha - \beta/2} . \mathrm{Var}(x) \tag{46}$$

$$\mathrm{Var}(T.\hat{x}) = \frac{\beta^2/\alpha}{2 - \alpha - \beta/2} . \mathrm{Var}(x) \tag{47}$$

$$\mathrm{Covar}(x, T.\hat{x}) = \frac{\beta(1 - \beta/2\alpha)}{2 - \alpha - \beta/2} . \mathrm{Var}(x) \tag{48}$$

Note that these values are marginally smaller than those for the predicted variable (except for identical velocity variance) as would be expected from general statistical considerations.

6.4 Covariance - third order

Haber(ref.4) has succeeded in deriving an exact relation for the covariances of the third order filtered variable, for the critically damped case, (although there is an error in one of them). Correct results are listed below. He uses an ingenious matrix factorization, which reduces the algebraic manipulations to a reasonable level. (Note that Haber's "α" is the "ρ" of this report, Section 4.4).

$$\mathrm{Var}(\hat{x}) = \frac{\rho\{66 - 186\rho + 202\rho^2 - 100\rho^3 + 19\rho^4\}. \mathrm{Var}(x)}{(2-\rho)^5} \tag{49}$$

$$\mathrm{Var}(T.\hat{x}) = \frac{\rho^3\{112 - 148\rho + 49\rho^2\}. \mathrm{Var}(x)}{2(2-\rho)^5} \tag{50}$$

$$\text{Var}(0.5T^2.\ \hat{\ddot{x}}) = \frac{3\rho^5 .\text{Var}(x)}{2(2-\rho)^5} \tag{51}$$

6.5 Summary of noise rejection

Table 2 gives approximate values of the ratio of variance of $\hat{X}$ to the variance of x, derived from equations (46) to (51), in terms of the basic real root, ρ, for critical and optimum damping, as defined in Sections 4.4, 4.5 and 4.6.

As would be expected from the statistical theory of variance, the trackers with more degrees of freedom (higher order) show the smallest reduction in noise. Thus the system order should be as small as consistent with acceptable lags with likely targets.

Note that in each case, the effective time constant τ of the tracking filter is given by T/ρ.

7. EQUIVALENT CONTINUOUS DATA TRACKER

Figure 7 shows a typical third order continuous-data tracking filter-predictor consisting of a velodyne integrator (loop A_3 - B) preceded by two integrators, A_2 and A_1, stabilised with resistors in series with the capacitors. These provide the well known proportional plus integral, error feedback. In the ideal case, with infinite amplifier gains:

$$\left.\begin{aligned}
\hat{\ddot{x}}_t &= \hat{\ddot{x}}_0 + \frac{1}{\tau_1}\int_0^t \hat{x}_c.dt + \hat{x}_c \\
\hat{\dot{x}}_t &= \hat{\dot{x}}_0 + \frac{1}{\tau_2}\int_0^t \hat{\ddot{x}}.dt + \ddot{x} \\
\hat{x}_t &= \hat{x}_0 + \frac{1}{\tau_3}\int_0^t \hat{\dot{x}}.dt
\end{aligned}\right\} \tag{52}$$

where:

$$\tau_1 = R.C_1$$

$$\tau_2 = R.C_2$$

$$\tau_3 = B$$

Taking the Laplace transform of each equation:

$$\left.\begin{aligned}
\bar{\hat{\ddot{x}}} &= \frac{\hat{\ddot{x}}_0}{s} + \bar{\hat{x}}_c.(1 + \frac{1}{s.\tau_1}) \\
\bar{\hat{\dot{x}}} &= \frac{\hat{\dot{x}}_0}{s} + \bar{\hat{\ddot{x}}}.\ (1 + \frac{1}{s.\tau_2}) \\
\bar{\hat{x}} &= \frac{\hat{x}_0}{s} + \frac{\bar{\hat{\dot{x}}}}{s.\tau_3}
\end{aligned}\right\}$$

Multiplying the first and second equations by $0.5T^2$ and T, respectively, to preserve the same form of vector used previously (T has no other significance, and cancels in practice when equations are solved):

$$\begin{bmatrix} 1 & -\dfrac{1}{T.s.\tau_3} & 0 \\ 0 & 1 & -\dfrac{1+s.\tau_2}{0.5T.s\tau_2} \\ 0 & 0 & 1 \end{bmatrix} .\bar{\hat{X}} = \frac{\hat{X}_0}{s} + \begin{bmatrix} 0 \\ 0 \\ \dfrac{0.5T^2(1+s.\tau_1)}{s.\tau_1} \end{bmatrix} .\bar{\hat{x}}_c$$

By multiplying through, by the left inverse of the left-most matrix:

$$\bar{X} = \begin{bmatrix} 1 & \dfrac{1}{T.s.\tau_3} & \dfrac{1+s.\tau_2}{0.5T^2.s^2.\tau_2.\tau_3} \\ 0 & 1 & \dfrac{1+s.\tau_2}{0.5T.s.\tau_2} \\ 0 & 0 & 1 \end{bmatrix} .\frac{\hat{X}_0}{s} + \frac{1}{s^3.\tau_1.\tau_2\tau_3} \begin{bmatrix} (1+s.\tau_1)(1+s.\tau_2) \\ (1+s.\tau_1)(1+s.\tau_2).T.s.\tau_3 \\ (1+s.\tau_1)0.5T^2.s^2.\tau_2.\tau_3 \end{bmatrix} .\bar{x}_c \qquad (53)$$

A comparison of equations (47) and (14), (15) and (16) shows the basic similarity of form, although minor differences exist. The transfer function between $\hat{x}$ and $\hat{x}_c$ can be made identical by making:

$$\left.\begin{aligned} \tau_3 &= \frac{T}{\alpha+\beta+\gamma} \\ \tau_1, \tau_2 &= T.\,\frac{\beta+3\gamma \pm \sqrt{(\beta-\gamma)^2 - 8\alpha.\gamma}}{4\gamma} \\ \tau_1+\tau_2 &= T.\,\frac{\beta+3\gamma}{2\gamma} \\ \tau_1.\tau_2 &= T^2.\,\frac{\alpha+\beta+\gamma}{2\gamma} \\ \tau_1.\tau_2.\tau_3 &= T^3/2\gamma \end{aligned}\right\} \qquad (54)$$

For a second order filter, $\gamma = 0$ and $\tau_1 = \infty$, whence:

$$\tau_2 = \frac{\tau_1.\tau_2}{\tau_1+\tau_2} = T.\,\frac{\alpha+\beta}{\beta} \qquad (55)$$

Table 3 shows the values of τ_1, τ_2, τ_3 for the first second and third order filters considered previously. It happens that the configuration of Figure 7 is not suitable for producing a third order filter with three equal real roots, as the required values of τ_1 and τ_2 are complex. The values given in the table gives roots ρ, ρ and 0.25ρ. Thus one root gives four times the time constant of the other two.

The equations (54) combined with equations (46) to (48) provide a means of estimating the noise rejection of the continuous data filter, fed with data with correlation time less than T. Thus, for a pulse radar, the time, T, corresponds to the inter-pulse period.

8. CONCLUSION

Basic equations have been derived for the third order exponential tracking filter-predictor operation and noise rejection. Normalised curves are given for the transient decay following position, velocity and acceleration steps, for critically damped first, second and third order filters, and for an optimally damped second order filter. These apply exactly, when the sampling interval is small compared with the filter time constant, and are fair approximations for ratios of up to 0.2. Performance for other values can be obtained exactly by digital computer.

The second order optimally damped filter gives best results providing the acceleration lags can be tolerated. This depends on the expected variation with time of the primary variable.

A third order, continuous data (analogue type) of tracking filter has also been examined, and equivalence relations with the exponential filter have been derived. This permits direct calculation of the noise rejection for analogue trackers, based on the decorrelation time of the input correction data.

In both continuous and sampled digital systems, the noise improvement is proportional to the equivalent number of uncorrelated correction estimates received per filter time constant; the lower the order of the filter the better the improvement.

NOTATION

D	discriminator function (Figure 1)
F	column vector (equation 10)
F_1	column vector (equation 22)
I	unit matrix
M	order of filter
P	prediction matrix of filter (equation 3)
P_x	prediction matrix of input variable
Q	general iterative matrix
R	general iterative column vector
T	intersample period
V	covariance generating matrix (equation 42)
X	input variable vector
X_0	initial condition vector
X_c	correction vector
a_0	initial acceleration of input variable
j	imaginary operator
q	element of Q
r	element of R
s	Laplace operator
t	time
v_0	initial velocity of input variable
x	element of X
x_0	initial position of input variable
x_c	correction in position (element of X_c)
Δ	characteristic equation (26)
α	filter parameter
β	filter parameter
γ	filter parameter
δ	noise sample
ρ	root of Δ
τ	filter time constant (T/ρ)
τ_1, τ_2, τ_3	time constants of analogue tracker
$\cdot$	differentiation with respect to time
$\wedge$	estimated value of .....
$-$	Laplace transform of .....
$t/t-1$	suffix indicating value of t, predicted at time (t-1)

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TABLE 1. ESTIMATE STEADY LAGS DUE TO TARGET MOTION

Order of system	α	β	γ	Position lag	Velocity lag	Acceleration lag
1st order	ρ	-	-	$v_0 . \tau$	-	-
2nd order critical	$\rho(2-\rho)$	ρ^2	-	$a_0 . \tau^2$	$a_0 . \tau . (2-\rho/2)$	-
2nd order optimum	$\frac{2\rho}{1+\rho}$	$\frac{2\rho^2}{1+\rho}$	-	$a . \tau^2 . (\frac{1+\rho}{2})$	$a_0 . \tau . (1+\rho/2)$	-
3rd order critical	$1-(1-\rho)^3$	$\frac{3\rho^2(2-\rho)}{2}$	$\rho^3/2$	0	0	0

TABLE 2. NOISE REJECTION PERFORMANCE OF TRACKERS

Order of system	α	β	γ	$Var(\hat{x})/Var(x)$	$Var(T.\hat{\dot{x}})/Var(x)$	$Var(0.5T^2.\hat{\ddot{x}})/Var(x)$
1st order	ρ	-	-	$0.5\rho(1+0.5\rho)$	-	-
2nd order critical	$\rho(2-\rho)$	ρ^2	-	$1.25\rho(1+0.1\rho)$	$0.25\rho^3(1+1.5\rho)$	-
2nd order optimum	$\frac{2\rho}{1+\rho}$	$\frac{2\rho^2}{1+\rho}$	-	$1.5\rho(1-0.67\rho)$	ρ^3	-
3rd order critical	$1-(1-\rho)^3$	$\frac{3\rho^2(2-\rho)}{2}$	$\rho^3/2$	$2.06\rho(1-0.32\rho)$	$1.75\rho^3(1+1.2\rho)$	$0.0469\rho^5(1+2.5\rho)$

TABLE 3. PARAMETERS OF CONTINUOUS FILTER-PREDICTOR

Order of system	τ_1	τ_2	τ_3
First order	-	-	T/ρ
Second order (critical)	-	$\frac{2T}{\rho}$	$\frac{T}{2\rho}$
Second order (optimum)	-	$\frac{T(1+\rho)}{\rho}$	$\frac{T}{2\rho}$
Third order ($\rho,\rho,0.25\rho$)	$3T/\rho$	$3T/\rho$	$T/2.25\rho$

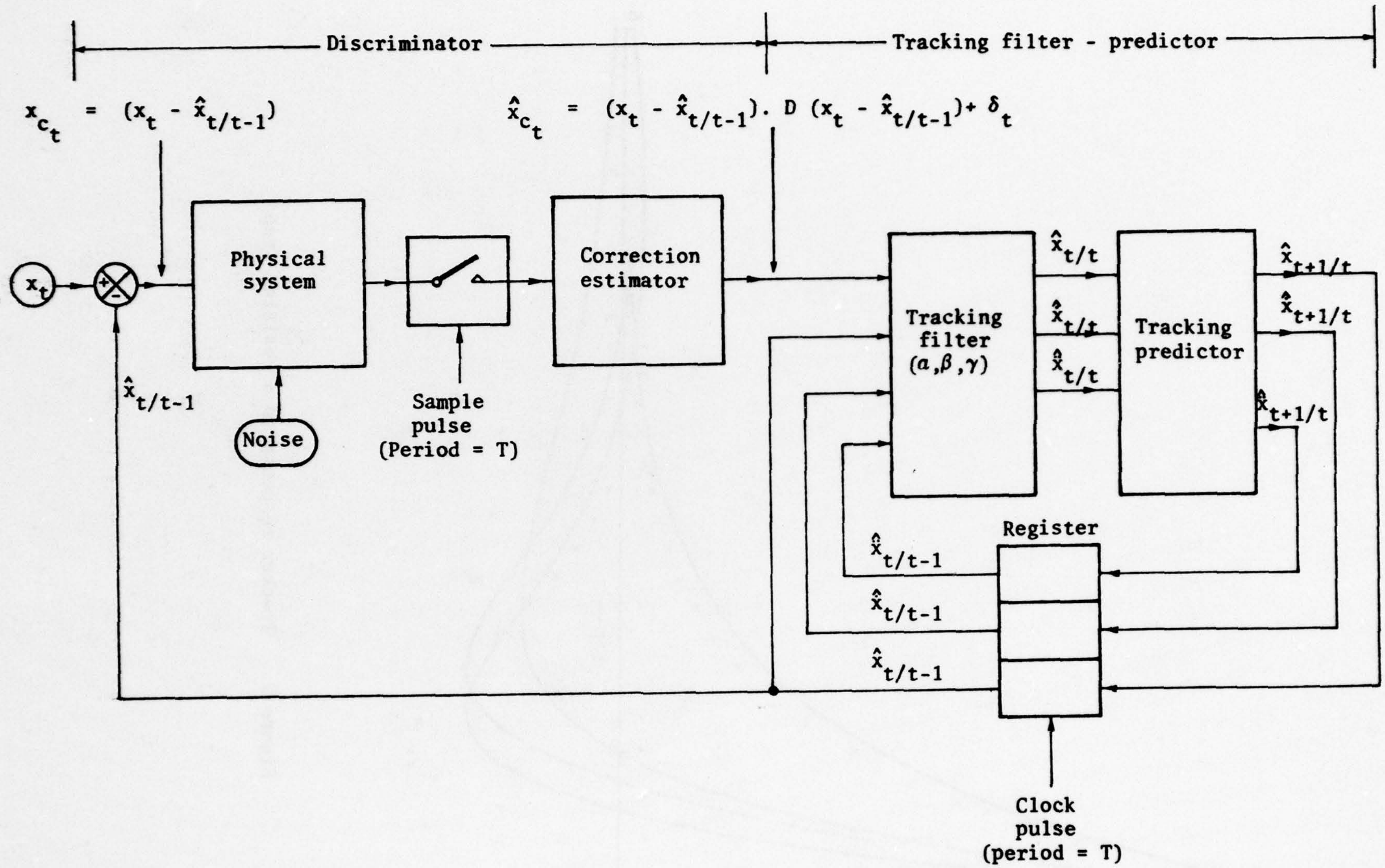

Figure 1. Sampled data tracking loop configuration

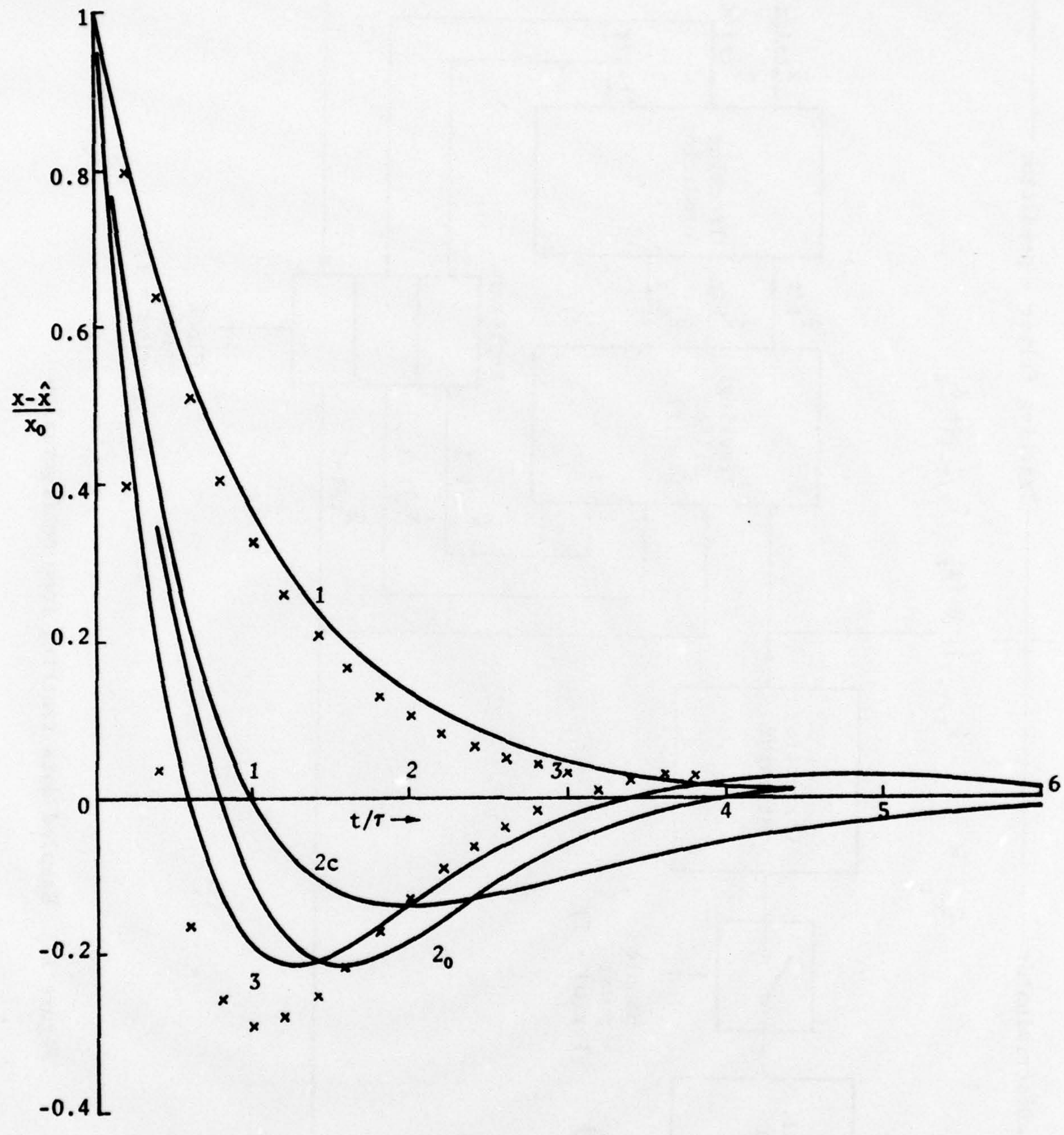

Figure 2. Tracker response to a position step

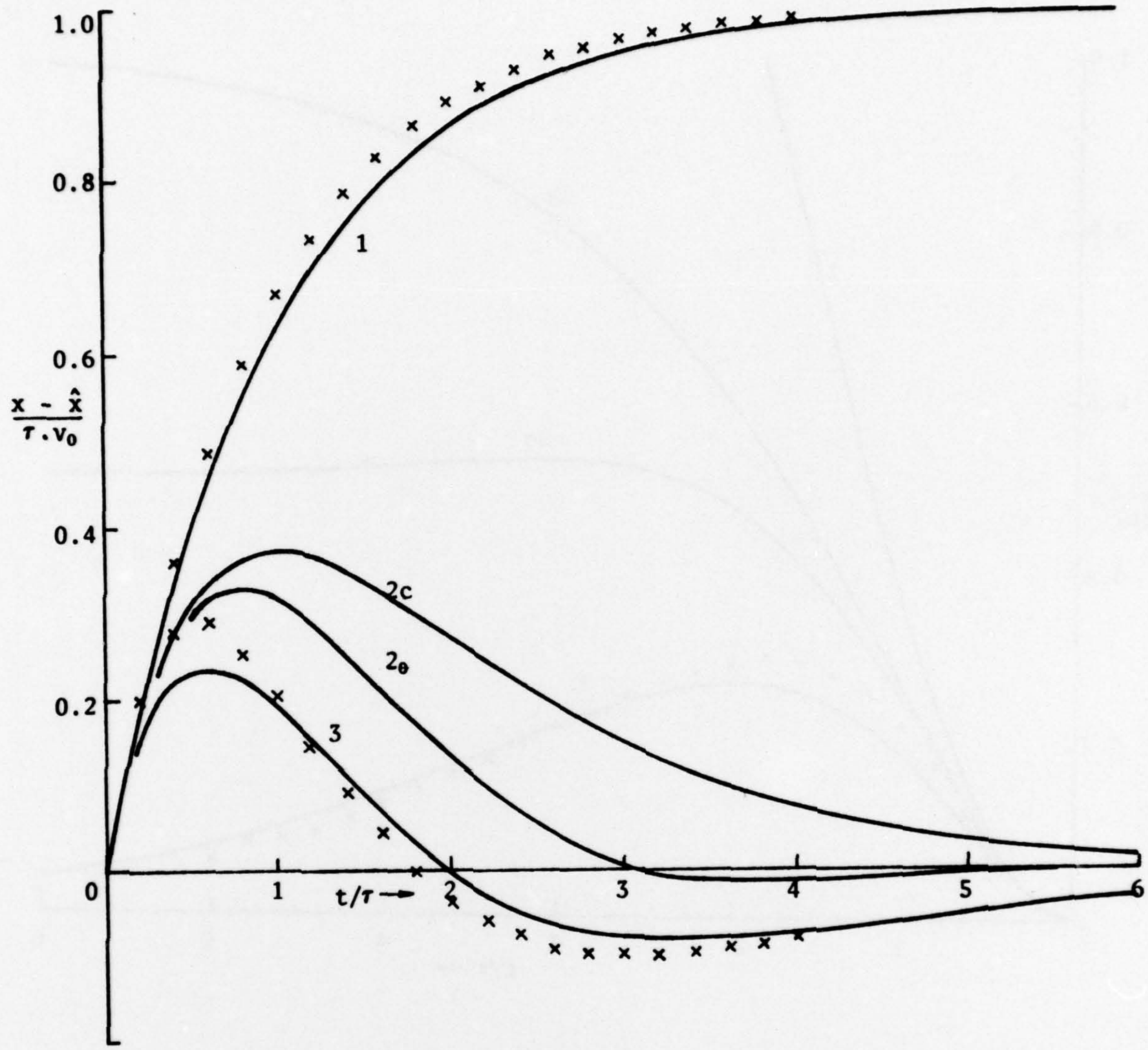

Figure 3. Tracker response to a velocity step

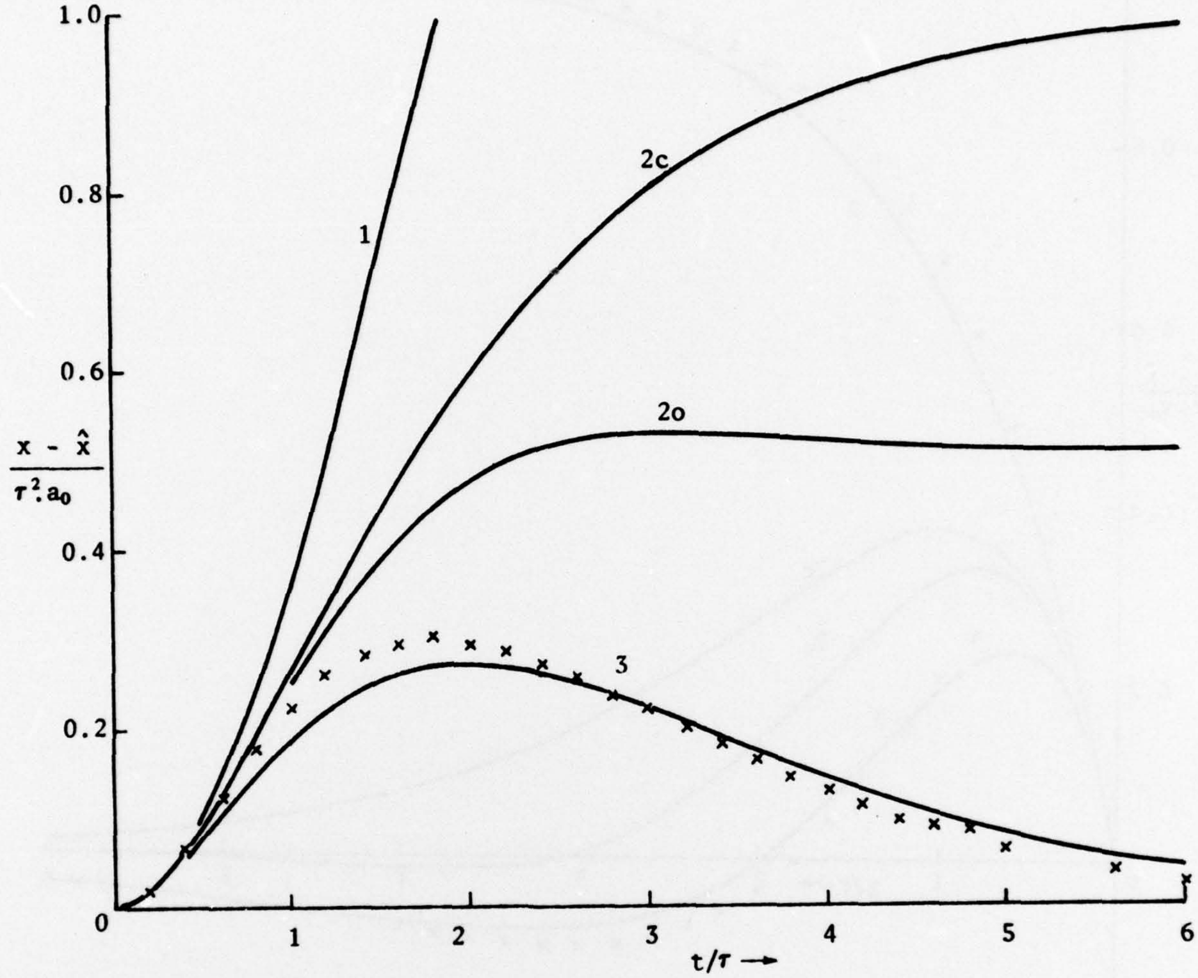

Figure 4. Tracker response to an acceleration step

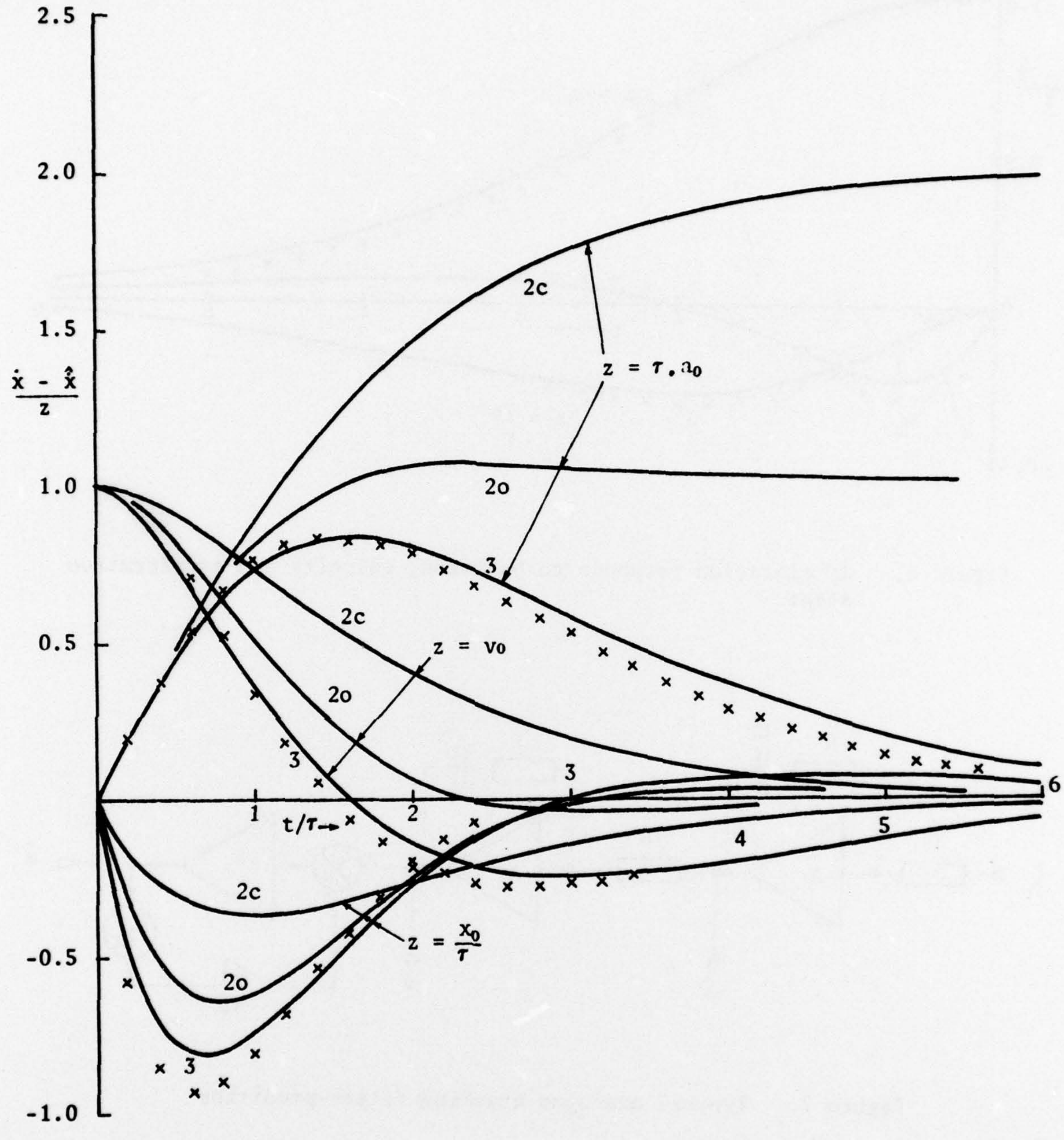

Figure 5. Velocity response to position, velocity and acceleration steps

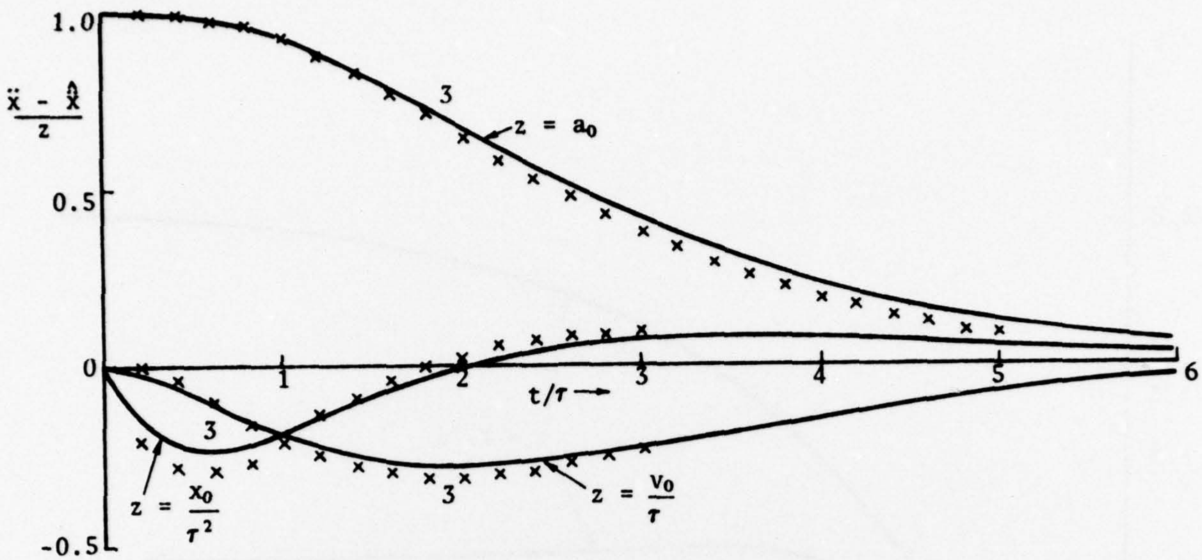

Figure 6. Acceleration response to position, velocity and acceleration steps

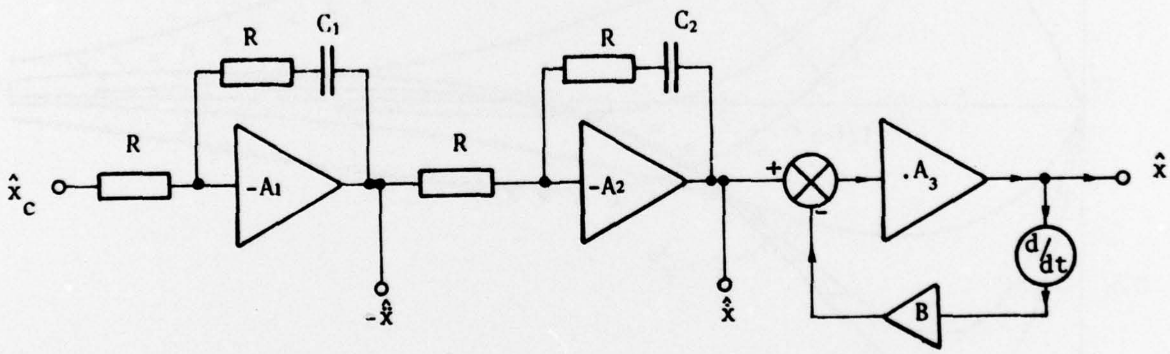

Figure 7. Typical analogue tracking filter-predictor

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